सत्यं शिवमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Major 2nd Semester Examination, 2024

MTMDSC202T-MATHEMATICS (MAJOR)

Time Allotted: 2 Hours

Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5=10$

(a) Show that the function $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined by 1+1

$$f(x)=x\sin\frac{1}{x}\ ,\quad x\neq 0$$
$$=\quad 1\quad ,\quad x=0$$

is not continuous at $x=0$. How can f be defined to be continuous at $x=0$?

(b) If $f(x,\,y)=\dfrac{xy}{x^2+y^2}$, $(x,\,y)\neq(0,\,0)$, show that $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=0$.

(c) State Taylor's theorem with Lagrange's and Cauchy's forms of remainder after n-terms. 1+1

(d) Justify the applicability of Rolle's Theorem for the function $f(x)=|x|,\ -1\leq x\leq 1$.

(e) Evaluate: $\displaystyle\int_0^{\pi/2}\sin^6 x\cos^4 x\,dx$

(f) From the definition of Gamma function $\Gamma(n)$, $n>0$, show that $\Gamma(n+1)=n\Gamma(n)$.

(g) Find the radius of curvature of the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ at one end of the minor axis.

(h) Find the envelope of the family of circles $(x-\alpha)^2+y^2=a^2$, α being the parameter.

2. (a) Using definition prove that $\displaystyle\lim_{x\to\infty}\frac{1}{x}\sin\frac{1}{x}=0$.

(b) Show that a function f differentiable at a point is continuous at that point.

(c) Let $f(x)$ be a real-valued function of a real variable x such that $f(x+y)=f(x)+f(y)$ for all $x,\,y\in\mathbb{R}$. If $f(x)$ is continuous at $x=0$, show that $f(x)$ is continuous everywhere in $\mathbb{R}$.

3. (a) If the function f is defined by 1

$$f(x)=x^2\sin\frac{1}{x}\ ,\quad x\neq 0$$
$$=\quad 0\quad ,\quad x=0$$

Show that f is differentiable at $x=0$ but not twice differentiable there.

(b) If $y = e^{m\sin^{-1}x}$, $|x| \le 1$, prove that 3

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0,$$

where y_n denotes the n-th derivative of y with respect to x.

(c) If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, $x \ne y$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$. 2

4. (a) Using Lagrange's Mean Value Theorem prove that 4

$$\frac{x}{1+x} < \log(1+x) < x \ , \quad x > 0$$

(b) State and prove Cauchy's Mean Value Theorem. 1+3

5. (a) Find the range of validity for the expansion of the function $\log(1+ax)$, when $a > 0$. 2

(b) Find the expansion of the function $\sin x$ about the point $x = \dfrac{\pi}{2}$. 3

(c) Determine a and b such that $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x) - b\sin x}{x^3} = 1$. 3

6. (a) Integrate: $\displaystyle\int \frac{e^{-x}\,dx}{e^x + 2e^{-x} + 3}$ 4

(b) Integrate: $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ 4

7. (a) If $I_n = \int \sec^n x\, dx$, then show that $(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$. 2+3

Hence evaluate: $\displaystyle\int_0^{\pi/4} \sec^5 x\, dx$

(b) Applying the definition of Beta function, evaluate $\displaystyle\int_0^{\pi/2} \cos^4 x\, dx$. 3

8. (a) Prove that the line $x\cos\alpha + y\sin\alpha = p$ will touch the curve $x^m y^n = a^{m+n}$ if $p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n\alpha \cos^m\alpha$. 4

(b) Find the asymptotes of the curve given by the equation: 4

$$x^2y^2 - a^2(x^2+y^2) - a^3(x+y) + a^4 = 0$$

9. (a) Find length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$. 3

(b) Find the volume of the solid obtained by the revolution of the curve $y^2(2a-x) = x^3$ about its asymptote. 5

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