सर्वं शिवमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours Part-I Examination, 2021

MATHEMATICS

PAPER: MTMA-II

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

GROUP-A

Answer any *three* questions from the following $5\times3 = 15$

1. State and prove Cantor's theorem on nested intervals. 1+4

2. (a) Let A and B be two non-empty bounded sets of real numbers and $C=\{x+y : x\in A, y\in B\}$. Show that $\sup C = \sup A + \sup B$. 2+3

   (b) Show that every bounded sequence has a convergent subsequence.

3. (a) For any subset $A\subset R$, prove that $(A')'\subset A'$ where A' denotes the set of all limit points of A. 3+2

   (b) For any two subsets $A, B\subset R$, show that the equality $(A\cap B)' = A'\cap B'$ does not hold in general.

4. (a) State Cauchy's second limit theorem. Using it find the limit 3+2

$$\lim_{n\to\infty}\frac{\{(n+1)(n+2)\ldots(2n)\}^{\frac{1}{n}}}{n}.$$

   (b) Evaluate : $\lim\limits_{x\to\infty}\frac{[x]}{x}$, if exists.

5. (a) Show that the sequence $\{x_n\}$ converges to 1 where 2+3

$$x_n=\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}.$$

   (b) Let A be a nonempty subset of R and $d(x, A)=\inf\{|x-y| : y\in A\}$. Prove that $d(x, A)=0$ if and only if $x\in\overline{A}$.

6. (a) Prove that a convergent sequence of real numbers is a Cauchy sequence. 2+3

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(b) Show that the sequence $\{x_n\}$ is not convergent where $x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, $n \geq 1$.

7. (a) Prove that union of two denumerable sets is denumerable. 2+3

(b) Prove that no nonempty proper subset of R is both open and closed in R.

8. (a) Let $D \subset R$ and f, g, h, be three function defined on D to R. Let $c \in D'$. If $f(x) \leq g(x) \leq h(x)$ for all $x \in D-\{c\}$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = l$ then prove that $\lim_{x\to c} g(x) = l$. 3+2

(b) Show that $\lim_{x\to 0} \sqrt{x}\ \sin\frac{1}{x} = 0$.

9. Let $t: R \to R$ be a continuous function and $f(x+y) = f(x)+f(y)$ for all $x, y \in R$. If $f(1) = k$ prove that $f(x) = kx$ for all $x \in R$. 5

GROUP-B

10. Answer any ***two*** questions from the following: 4×2 = 8

(a) Evaluate $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$. 4

(b) If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, prove that $I_n = \frac{1}{n-1} - I_{n-2}$ 2+2

Hence find the value of I_5.

(c) (i) Prove that $\frac{B(m, n+1)}{n} = \frac{B(m,n)}{m+n}$ 2+2

(ii) Evaluate $\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$.

11. Answer any ***three*** questions from the following: 4×3 = 12

(a) Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the parameters a, b are connected by the relation $a+b=c$, c being a nonzero constant.

(b) Find all the asymptotes of the curve $x^2y + xy^2 + xy + y^2 + 3x = 0$.

(c) Show that the curve $(x+y)^3 - \sqrt{2}\,(y-x+2) = 0$ has a double point at $(-1, 1)$. Find the equation of the tangents at that point and identify the nature of the double point.

(d) If ρ_1 and ρ_2 are the radii of curvature at the ends of conjugate diameter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then prove that $\rho_1^{2/3}+\rho_2^{2/3} = \frac{a^2+b^2}{(ab)^{2/3}}$.

(e) Determine the pedal equation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to a focus. Where $a>b$.

GROUP-C

Answer any *one* question from the following $10\times1=10$

12.(a) Examine whether the equation $(\cos y+y\cos x)\,dx+(\sin x-x\sin y)\,dy=0$ is exact or not and then solve it. 5

(b) Find the orthogonal trajectories of the cardiodes $r=a(1-\cos\theta)$. 5

13.(a) Reduce the equation $x^2(y-px)=p^2y$ to Clairaut's form by putting $x^2=u$ and $y^2=v$. Hence obtain the general and singular solution. 5

(b) Solve the following differential equation $y=(1+p)x+ap^2$. 5

14.(a) Solve by the method of undetermined coefficient: $(D^2+4)y=x^2\sin 2x$. 5

(b) Solve : $(x^2D^2-3xD+5)\,y=x^2\sin(\log x)$. 5

15.(a) Solve by the method of variant of parameters: 5

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{e^x}{1+e^x}.$$

(b) Solve by reducing to a linear equation: $(1+x^2)\frac{dy}{dx}-4x^2\cos^2 y+x\sin 2y=0$. 5

16.(a) Solve: $x^4\frac{d^3y}{dx^3}+3x^3\frac{d^2y}{dx^2}-2x^2\frac{dy}{dx}+2xy=\log x$. 5

(b) Solve: $\left(\frac{d^2y}{dx^2}+y\right)\cot x+2\left(\frac{dy}{dx}+y\tan x\right)=\sec x$ by reducing it to normal form. 5

17.(a) Solve by the method of operational factors: 5

$$x\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}-y=x^2.$$

(b) Solve: $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$ by changing the independent variable. 5

GROUP-D

Answer any *one* question from the following $5\times1=5$

18. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three unit vectors such that $\vec{a}\times\vec{b}\times\vec{c}=\frac{1}{2}\vec{b}$, find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$; $\vec{b}$, $\vec{c}$ being non parallel.

$a > b$.

GROUP-C

Answer any *one* question from the following $10\times1=10$

12.(a) Examine whether the equation $(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$ is exact or not and then solve it. 5

(b) Find the orthogonal trajectories of the cardiodes $r = a(1-\cos\theta)$. 5

13.(a) Reduce the equation $x^2(y-px) = p^2 y$ to Clairaut's form by putting $x^2 = u$ and $y^2 = v$. Hence obtain the general and singular solution. 5

(b) Solve the following differential equation $y = (1+p)x + ap^2$. 5

14.(a) Solve by the method of undetermined coefficient: $(D^2+4)y = x^2\sin 2x$. 5

(b) Solve : $(x^2D^2 - 3xD + 5)\,y = x^2\sin(\log x)$. 5

15.(a) Solve by the method of variant of parameters: 5

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}.$$

(b) Solve by reducing to a linear equation: $(1+x^2)\frac{dy}{dx} - 4x^2\cos^2 y + x\sin 2y = 0$. 5

16.(a) Solve: $x^4\frac{d^3y}{dx^3} + 3x^3\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 2xy = \log x$. 5

(b) Solve: $\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = \sec x$ by reducing it to normal form. 5

17.(a) Solve by the method of operational factors: 5

$$x\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} - y = x^2.$$

(b) Solve: $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$ by changing the independent variable. 5

GROUP-D

Answer any *one* question from the following $5\times1=5$

18. If $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $\vec{a}\times\vec{b}\times\vec{c} = \frac{1}{2}\vec{b}$, find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$; $\vec{b}$, $\vec{c}$ being non parallel.

19. Show by vector method, that the straight line joining the mid points of two non-parallel sides of a trapezium are parallel to the parallel sides and half of their sum in length.

20. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, prove that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

21.(a) Forces $\vec{P}$, $\vec{Q}$ act at O and have a resultant $\vec{R}$. If any transversal cuts lines of action of $\vec{P}$, $\vec{Q}$ and $\vec{R}$ at A, B, C respectively, then show that $\frac{|\vec{P}|}{OA}+\frac{|\vec{Q}|}{OB}=\frac{|\vec{R}|}{OC}$. 3

(b) A particle acted on by constant forces $4\hat{i}+5\hat{j}-3\hat{k}$ and $3\hat{i}+2\hat{j}+4\hat{k}$ is displaced from the point $\hat{i}+3\hat{j}+\hat{k}$ to the point $2\hat{i}-\hat{j}-3\hat{k}$. Find the total work done by the forces. 2

22.(a) Find the moment of the force $4\hat{i}+2\hat{j}+\hat{k}$ acting at a point $5\hat{i}+2\hat{j}+4\hat{k}$ about the point $3\hat{i}-\hat{j}+3\hat{k}$. 3

(b) Find the vector equation of the plane passing through the origin and parallel to the vectors $2\hat{i}+3\hat{j}+4\hat{k}$ and $4\hat{i}-5\hat{j}+4\hat{k}$. 2

23.(a) Find the constants a, b, c so that $\vec{V}=(x+2y+az)\,\hat{i}+(bx-3y-z)\,\hat{j}+(4x+cy+2z)\,\hat{k}$ is irrotational. 3

(b) Find a so that $\vec{V}=3x\hat{i}+(x+y)\hat{j}-ax\,\hat{k}$ is solenoidal. 2

24. Show that the necessary and sufficient condition that a non-zero vector $\vec{u}$ always remains parallel to a fixed line is that $\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}$.

25.(a) If $f(r)$ is differentiable, then prove that $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$. 3

(b) Show that $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=-\frac{\vec{a}}{r^3}+\frac{3(\vec{a}\cdot\vec{r})\vec{r}}{r^5}$, where $\vec{a}$ is a constant vector. 2

26. $\vec{V}=2xz^2\hat{i}-yz\hat{j}+3xz^3\hat{k}$ and $\phi=x^2yz$, then find

(a) curl $(\phi\vec{V})$ 3

(b) curl curl$\vec{V}$. 2

N.B.: *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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