WEST BENGAL STATE UNIVERSITY

B.Sc. Honours PART-III Examinations, 2017

MATHEMATICS-HONOURS

PAPER-MTMA-V

Time Allotted: 4 Hours

Full Marks: 100

The figures in the margin indicate full marks. Candidates should answer in their own words and adhere to the word limit as practicable. All symbols are of usual significance.

Group-A
(Marks-70)

Answer Question No. 1 and any *five* questions from the rest.

1. Answer any *five* questions from the following: $3\times5=15$

(a) Let A be an uncountable subset of $\mathbb{R}$ such that A^d (the derived set of A) is compact in $\mathbb{R}$. Does it follow that A is bounded in $\mathbb{R}$? Justify your answer.

(b) Find the radius of convergence of the power series

$$x+\frac{x^2}{2^2}+\frac{2!}{3^3}x^3+\frac{3!}{4^4}x^4+\ldots\ldots$$

(c) Examine if the series $\sum_{n=0}^{\infty}\frac{x^4}{(1+x^4)^n}$ is uniformly convergent on [0, 1].

(d) Defining $\log_e x=\int_1^x\frac{dt}{t}$, $(x>0)$

Prove that $\frac{x}{1+x}<\log_e(1+x)<x$, $(x>0)$.

(e) If $f : [a, b] \to \mathbb{R}$ be Riemann integrable in $[a, b]$, prove that $F(x) = \int_a^x f(t)dt;\ a \le x \le b$; is continuous in $[a, b]$.

(f) Find the intrinsic equation of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, when arc length is measured from a cusp on the x-axis.

(g) Prove that if $\int_a^{\infty} f$ be absolutely convergent, then f is convergent.

(h) Prove or disprove: The trigonometric series $\sum_n \frac{\sin nx}{n^2}$ represents a Fourior series in $[-\pi, \pi]$.

(i) Evaluate $\iint_R xy(x^2+y^2)dxdy$ over $R : [0, a; 0, b]$.

2. (a) If every infinite subset of $S(\subset\mathbb{R})$ has a limit point in S, prove that S is compact. 5

(b) If A and B are two open set in $\mathbb{R}$ such that $A \cap B$ is compact. Prove that $A \cap B = \Phi$. 3

(c) Prove or disprove: The range of any convergent sequence in $\mathbb{R}$ is a compact set. 3

3. (a) Let for each $n\in N$, $f_n : [a, b] \to R$ be Riemann integrable on $[a, b]$ and the sequence $\{f_n\}_n$ converges uniformly to f on $[a, b]$. Show that f is Riemann integrable on $[a, b]$, also show that $\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b [\lim_{n\to\infty} f_n]dx$. 3+3

(b) Show that the series $\sum_{n=1}^{\infty} \frac{n^4+1}{n^7+5}\left(\frac{x}{3}\right)^n$ is uniformly convergent on $[-3, 3]$. 2

(c) Prove, with proper justification, that $\lim_{x\to 0} \sum_{n=2}^{\infty} \frac{\cos nx}{n(n+1)} = \frac{1}{2}$. 3

4. (a) If $\{f_n\}_n$ is a sequence of real valued continuous functions defined on a closed set $D(\subset\mathbb{R})$ converging uniformly on D, then show that 4

$$\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x).$$

(b) Show that the sequence $\{f_n(x)\}_n$ where $f_n(x)=\dfrac{x}{1+nx^2}$; $0\le x\le 1$ converges uniformly to the limit function $f(x)$ in [0, 1] and prove that $f'(x)=\lim\limits_{n\to\infty} f_n'(x)$ is true when $x\ne 0$ but is not true when $x=0$. 4

(c) Let $\rho=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}$, $a_n\in\mathbb{R}$ for all n. Prove that the power series $\sum\limits_{n=1}^{\infty}a_n x^n$ is everywhere convergent if $\rho=0$, but the series nowhere convergent if $\rho=\infty$. 3

5\. (a) Establish the following relation between the Beta and Gamma function: $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$; $m,n>0$. 4

(b) By power series expansion of $(1-x^2)^{-\frac{1}{2}}$, derive the power series of $\sin^{-1}x$. Hence show that 4

$$\frac{\pi}{2}=1+\frac{1}{2}.\frac{1}{3}+\frac{1}{2}.\frac{3}{4}.\frac{1}{5}+\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{1}{7}+\ldots$$

(c) Show with proper justification, that 3

$$\int_0^{\infty} e^{-x^2}\cos\alpha x\,dx=\frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}},\ \text{for all real } \alpha.$$

6\. (a) Let $f:[a,b]\to R$ be a bounded function. Let S be the set of all points of discontinuity of f. If S is a finite set, show that f is Riemann integrable on $[a,b]$. 5

(b) Let f be bounded and integrable on $[a,b]$ and there exist a function F such that $F'(x)=f(x)$ for all $x\in[a,b]$. Prove that $\int_a^b f(x)dx=F(b)-F(a)$. 4

(c) Define primitive of a function. Give an example of a Riemann integrable function without any primitive. 2

7\. (a) Let $f:[a,b]\to\mathbb{R}$ be a bounded function and M and m respectively be the lub and glb of f in $[a,b]$. Let P be a partition over $[a,b]$ with $\|P\|<\delta$ and Q be a refinement of P having k more points of division than that of P. Prove that $U(Q,f)\le U(P,f)\le U(Q,f)+(M-m)\delta k$. 4

(b) Let $f: [0, 1] \to \mathbb{R}$ defined by 3

$$f(x)=\begin{cases} 1, & \text{if } x \text{ rational} \\ 0, & \text{if } x \text{ irrational} \end{cases}$$

Examine whether f is Riemann integrable over [0, 1] or not.

(c) Prove that $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x+\cos x}dx < \frac{\pi^3}{24}$. 4

8. (a) The function f: $[a, b] \to R$ obeys Lipschitz's condition on $[a, b]$. Prove that f is of bounded variation. Is the converse True? Support your answer. 4

(b) Let $f, g : [0, 1] \to R$ defined by 4

$$f(x)=\begin{cases} x^3 \sin\frac{1}{x^2}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

$$g(x) = 3x^2 + \cos x.$$

Examine whether the curve $\gamma = (f, g)$ is rectifiable.

(c) Using Lagrange's method find the points on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ whose distance from the line $3x + y = 9$ are least and greatest. 3

9. (a) State and prove Taylor's Theorem for a real valued function of two variables. 1+4

(b) If $f(x, y) = \sin\pi x + \cos\pi y$, use Mean-value theorem to express $f\left(\frac{1}{2},0\right)-f\left(0,-\frac{1}{2}\right)$ in terms of first order partial derivative of f and deduce that there exist θ in (0, 1) such that $\frac{4}{\pi}=\cos\frac{\pi}{2}\theta+\sin\frac{\pi}{2}(1-\theta)$. 3

(c) Prove or disprove: A real valued function defined on an open set G in $\mathbb{R}^2$ having both first order partial derivative zero at some point z_0 in G has a local extremum at the point z_0. 3

10.(a) Show that the function $f(x)=|x|$ satisfies Dirichlet's conditions in $[-\pi, \pi]$. Obtain the Fourier series of $f(x)$ in $[-\pi, \pi]$ and show that $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ 5

(b) Find the value of $\iiint\limits_E \frac{dx\,dy\,dz}{(1+x+y+z)^3}$, where E is the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$. 3

(c) Find the area of surface of the cylinder $x^2+y^2=4a^2$ above the xy plane and bounded by the planes $y=0$, $z=a$, $y=z$. 3

Group-B

[Marks-15]

Answer any *one* question from the following. $15\times1=15$

11.(a) Let X be the set of all real valued continuous functions on $[0, 1]$ and $d\colon X\times X\to\mathbb{R}$ be defined as $d(x,y)=\int_0^1 |x(t)-y(t)|\,dt$, for $x, y\in X$. 4+3

Prove that d defines a metric on X.

Also show that for each $n\in N$, the function $x_n : [0, 1]\to\mathbb{R}$ defined by

$$x_n(t)=\begin{cases} n, & \text{if } \ 0\le t\le \dfrac{1}{n^2} \\[2mm] \dfrac{1}{\sqrt{t}}, & \text{if } \ \dfrac{1}{n^2}<t\le 1 \end{cases}$$

belongs to X and $\{x_n\}$ is a Cauchy sequence in (X, d).

(b) If $\{x_n\}$ is a sequence in a metric space (X, d) and $x\in X$. Prove that x is a cluster point of $\{x_n\}$ if and only if there exists a subsequence of $\{x_n\}$ converging to x. 4

(c) Prove that in a metric space (X, d) every open ball is an open set. Is the converse true? – Support your answer. 4

12.(a) Prove that the function $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by 4

$$d(\alpha, \beta) = |x_1 - y_1| + |x_2 - y_2|, \text{ for } \alpha, \beta \in \mathbb{R}^2$$

where $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$ defines a metric on $\mathbb{R}^2$.

(b) Prove that in a complete metric space (X, d), for any descending sequence $\{F_n\}$ of non-empty closed sets such that diam. $(F_n) \to 0$ as $n \to \infty$, $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point. 4

(c) Give the definition of a bounded sequence in a metric space. 1+3+1+2
Prove that a convergent sequence in a metric space is bounded.
Is a bounded sequence in a metric space always convergent? Is a bounded sequence in a metric space a Cauchy sequence? – Support your answer.

Group-C
[Marks-15]

Answer any *one* question from the following. $15 \times 1 = 15$

13.(a) If $Z = (x_1, x_2, x_3)$ is the projection on the Riemann sphere of the point $z = x + iy$ in the complex plane, then show that 3+2

$$x_1 = \frac{2\operatorname{Re} z}{|z|^2 + 1}, \quad x_2 = \frac{2\operatorname{Im} z}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1},$$

where the Riemann sphere is given by $x_1^2 + x_2^2 + x_3^2 = 1$.

Find the projection of $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$ on the Riemann sphere.

(b) Let $f(z) = u(x, y) + iv(x, y)$ be defined on some open set G containing the point z_0. If the first partial derivative of u and v, that exist in G, are continuous at z_0 and satisfy the Cauchy-Riemann equations at z_0 then prove that f is differentiable at z_0. 5

(c) Show that $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and use Milne-Thomson method to find an analytic function whose real part is given by $u(x, y)$. 5

14.(a) Prove that a function $f: G \to \mathbb{C}$ is continuous at $z_0 \in G$ if and only if for any sequence $\{z_n\}$ in G converging to z_0, the sequence $\{f(z_n)\}$ converges to $f(z_0)$. 4

(b) Prove that the function $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \text{Re } z$, for $z \in \mathbb{C}$ is continuous everywhere but is differentiable nowhere on $\mathbb{C}$. 5

(c) Let $f: \mathbb{C} \to \mathbb{C}$ be defined as 6

$$f(z) = \begin{cases} \dfrac{x^{4/3} y^{5/3} + i x^{5/3} y^{4/3}}{(x^2 + y^2)}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

Show that f satisfies Cauchy-Riemann equations at $z = 0$ but is not differentiable at this point.