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लक्ष्यं विद्यमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours PART-I Examinations, 2017

MATHEMATICS-HONOURS

PAPER-MTMA-II

Time Allotted: 4 Hours | Full Marks: 100

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Group-A

Answer any *five* questions from the following: $5\times5=25$

1. (a) Using order completeness property of R prove that for every real number x, there is a positive integer n such that $n > x$. 2

   (b) Prove that the set Q of rational numbers is dense and Archimedean. 1+2

2. (a) Prove that closure of a set is a closed set. 2

   (b) If F is a closed set in R then prove that the complement of F is an open set in R. 2

   (c) Find the derived set of the set Q of rational numbers. 1

3. (a) Prove that a convergent sequence is bounded. 2

(b) If $x_n = \left(1+\frac{1}{n}\right)^n$, show that the sequence $\{x_n\}$ is monotonically increasing and bounded above. State with reasons whether it is convergent or not. 1+1+1

4. (a) If a sequence $\{x_n\}$ converges to l, then prove that $\{|x_n|\}$ converges to $|l|$. Is the converse true? Give reasons. 2+1

(b) Show that $\left\{\frac{n}{n+1}\right\}$ is a Cauchy sequence and $\{2^n\}$ is not a Cauchy sequence. 1+1

5. (a) Use Sandwich theorem to show that the sequence $\{x_n\}$ where $x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$ converges to 0. 2

(b) If $\{x_n\}$ converges to l, then prove that $\{y_n\}$ converges to l where $y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$. 3

6. (a) Prove that an infinite subset of an enumerable set is enumerable. 3

(b) Show that the set of rational numbers is enumerable. 2

7. (a) Verify Bolzano-Weierstrass theorem for the set $\left\{\frac{n}{n+1} : n \in N\right\}$. 2

(b) Prove or disprove: Boundedness is a necessary condition for a set to have a limit point. 1

(c) For any positive real numbers p and a, prove that 2

$$\lim_{n\to\infty} \frac{n^p}{(1+a)^n} = 0.$$

8. (a) If $\lim_{x \to a} f(x) = l(\neq 0)$, then prove that there exists a neighborhood of a where $f(x)$ and l will have the same sign. 2

(b) Discuss the continuity of the function 2

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

at $x = 0$.

(c) Give an example of discontinuity of second kind. 1

9. (a) If $f : D \to R$ be a function continuous at $x = c$, $c \in D$ then prove that for every sequence $\{x_n\}$ in D converging to c, $\{f(x_n)\}$ converges to $f(c)$. 2

(b) If $f(x) = \begin{cases} x, & x \in Q \\ -x, & x \in R - Q \end{cases}$ 3

then prove that $x = 0$ is the only point of continuity.

Group-B

10. Answer any *two* questions from the following: $4 \times 2 = 8$

(a) Let $I_n = \int_0^1 x^n \tan^{-1} x \, dx$; $n > 2$, $n \in \mathbb{N}$, then show that

$$(n+1)I_n - (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

(b) Show that, $2^{2m-1}\,\Gamma(m)\,\Gamma(m + \tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2m)$, $m > 0$.

(c) Evaluate: $\displaystyle\int_0^{\pi/2} \frac{dx}{(c^2 \cos^2 x + b^2 \sin^2 x)^2}$; $a, b > 0$.

35
12
47

11. Answer any *three* questions from the following: 4×3 =

(a) Find the asymptotes of

$$x^2(x^2-y^2)(x-y)+2x^3(x-y)-4y^3=0.$$

(b) Show that the pedal of the circle $r=2a\cos\theta$ w.r. to origin is the cardioid $r=a(1+\cos\theta)$.

(c) Show that the origin is a double point of the curve $x^3=y^3+ay^2$, $a\neq 0$ and it is a single cusp of first species.

(d) Find the evolute of the curve

$$x=a(\cos t+t\sin t);\ y=a(\sin t-t\cos t).$$

(e) If ρ_1 and ρ_2 are radii of curvatures at two extremities of any chord of the cardioid $r=a(1+\cos\theta)$ passing through the pole, prove that $\rho_1^2+\rho_2^2=\dfrac{16a^2}{9}$.

$\frac{28+2}{37}$ $\frac{37+5}{52}$

Group-C

Answer any *three* questions from the following 10×3 = 30

12.(a) Find the integrating factor and hence solve: 5

$$(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0.$$

(b) Find the orthogonal trajectories of the family of coaxial circles 5

$$x^2+y^2+2gx+c=0,$$

where g is the parameter and c is a constant.

13.(a) Reduce the equation $xp^2-2py+x+2y=0$ to Clairaut's form by putting $x^2=u$ and $y-x=v$. Hence obtain the general and singular solution. 5

(b) Find the general and singular solution of the equation 5

$$y = px + \sqrt{a^2p^2 + b^2}\,.$$

$1+e^x = z$
$e^x dx = dz$

14.(a) Solve by the method of undetermined coefficient: 5

$$(D^2 - 3D)y = x + e^x \sin x\,.$$

(b) Solve: $(D^2 + 2D + 1)y = x\cos x$. 5

$-\int \frac{e^{-x}}{1+e^{x}}\,dx$

$= -\int \frac{-dz}{z}$

15.(a) Solve by the method of variation of parameters: 5

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}\,.$$

$1+e^x = z$
$e^x dx = dz$

(b) Solve: $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x\log x$. 5

16.(a) Show that $x\frac{d^3y}{dx^3} + (x^2+x+3)\frac{d^2y}{dx^2} + (4x+2)\frac{dy}{dx} + 2y = 0$ is exact and solve it. 5

(b) Solve: $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x\sec x$ by reducing it to normal form. 5

17.(a) Solve by the method of operational factors: 5

$+1-4$
$+3+1$

$$(x+3)\frac{d^2y}{dx^2} - (2x+7)\frac{dy}{dx} + 2y = (x+3)^2e^x\,.$$

$+1+2-3+1$

(b) Solve: $x^6\frac{d^2y}{dx^2} + 3x^5\frac{dy}{dx} + 9y = \frac{1}{x^2}$ by changing the independent variable. 5

$-1+2+3+1$

$1-4-3+2$

Group-D

Answer any *five* questions from the following $5\times5=25$

18.(a) Find the equation of plane passing through the points whose position vectors are $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+3\hat{j}+2\hat{k}$. 3

(b) Find an unit vector perpendicular to both the vectors $2\hat{i}-6\hat{j}-3\hat{k}$ and $4\hat{i}+3\hat{j}-\hat{k}$. 2

19. Using vector method prove the formula 5

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

20. For three vectors $\vec{a},\vec{b},\vec{c}$ prove that 5

$$[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}]=[\vec{a}\vec{b}\vec{c}]^2.$$

21.(a) Let $ABCD$ be a parallelogram. Prove by vector method $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2$. 3

(b) Find the work done when a force $\vec{F}=4\hat{i}-3\hat{j}+2\hat{k}$ applied to an object and it moves from the point (3, 2, –1) to the point (2, –1, 4) along a straight line. 2

22. A force $\vec{F}=3\hat{i}+2\hat{j}-4\hat{k}$ is applied at a point (1, –1, 2). Find the moment of the force about the point (2, –1, 3). 5

23.(a) Show that the parallelograms on the same base and between the same parallel lines are equal in area. 3

(b) Find the directional derivative of $\phi=x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction $\overrightarrow{PQ}$ where $Q(5,0,4)$. 2

24.(a) If $\phi = \ln r$, show that $\vec{\nabla}\phi = \frac{\vec{r}}{r^2}$. 2

(b) Prove that for a scaler field ϕ, $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$. 3

25. Prove that if a vector $\vec{v}$ has continuous second order partial derivatives, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$. 5

26. Define an irrotational vector. Find the constants a, b, c such that the vector $\vec{V} = (-4x-3y+az)\hat{i} + (bx+3y+5z)\hat{j} + (4x+cy+3z)\hat{k}$ is irrotational. 1+4

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