WEST BENGAL STATE UNIVERSITY

B.Sc. Honours PART-III Examination, 2016

MATHEMATICS-HONOURS

Paper-MTMA-V

Time Allotted: 4 Hours | Full Marks: 100

The figures in the margin indicate full marks. Candidates should answer in their own words and adhere to the word limit as practicable. All symbols are of usual significance.

Gourp-A

[Marks-70]

Answer Question No. 1 and any *five* from the rest.

1. Answer any *five* questions: $3\times5=15$

(a) If $A=\left(-\frac{1}{2},\frac{1}{2}\right)$ and $B=\left\{\pm\frac{1}{2},\pm 2\right\}$, then examine whether $A\cup B$ is compact in $\mathbb{R}$.

(b) Justify: If $\sum_{n=1}^{\infty} x_n$, $(x_n>0)$ is convergent then $\sum_{n=1}^{\infty} x_n^2$ is convergent.

(c) Let $f_n(x)=\frac{\sin nx}{\sqrt{n}}$, $x\in[-1, 1]$. Does $\{f_n\}$ converge uniformly on $[-1, 1]$? Justify.

(d) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = x \sin \frac{1}{x}, \text{ for } x \neq 0$$
$$= 0 \quad , \text{ for } x = 0$$

is not of bounded variation on $[0, 1]$.

(e) Assuming $\Gamma(m)\,\Gamma(1-m) = \pi \operatorname{cosec} m\pi$, $0 < m < 1$
Show that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$.

(f) Verity whether the value of the integral $\int_0^3 x\,d([x]-x)$ is $\frac{3}{2}$ (where $[x]$ denotes the greatest integer not greater than x).

(g) Prove or disprove: If $|f|$ is Riemann integrable over a closed and bounded interval I, then f in also Riemann integrable over I.

(h) If e is defined by the equation $\int_1^e \frac{dt}{t} = 1$, prove that $2 < e < 3$.

(i) Appling Dirichlet's test determine the convergence of $\int_1^\infty \sin x^2\,dx$.

2. (a) (i) Give an example of a (1+1)+2

(a) closed subset of $\mathbb{R}$ which in not compact

(b) bounded subset of $\mathbb{R}$ which is not compact

(ii) For a closed subset S and compact subset T of $\mathbb{R}$, show that $S \cap T$ is compact.

(b) Prove that every compact of $\mathbb{R}$ is closed and bounded. 3

(c) Show that a real valued continuous function on a closed and bounded subset of $\mathbb{R}$ is uniformly continuous. 4

3. (a) When is a sequence of functions $f_n : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$, $n \in N$, said to uniformly converge to a function $f : S \to \mathbb{R}$? 1+3

Let $\{f_n : S \to \mathbb{R}, S \subseteq \mathbb{R}, n \in N\}$ converges uniformly on S. If each f_n is continuous at a point c of S, then show that the limit function f in also continuous at c.

(b) If a sequence of functions $\{f_n: S \to \mathbb{R}, S \subseteq \mathbb{R}, n \in N\}$ satisfies $|f_n(x)| \le M_n$ ($x \in S, n = 1, 2, \ldots..$), 4

Prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly if $\sum_{n=1}^{\infty} M_n$ converges.

(c) Let $f_n(x) = n^2x(1-x)^n$, $x \in [0, 1]$, $n \in N$. 3

Verity whether $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$

4. (a) State Abel's test of uniform convergence. Using this, 1+3

Show that $\sum_{n\to 1}^{\infty} \frac{(-1)^n}{n^{x+\frac{1}{2}}}$ converges uniformly on $[0, \infty]$.

(b) Let the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ be r. 3

Find the radius of convergence of $\sum_{n=0}^{\infty} a_n x^{2n}$.

(c) Assuming the power saris expansion for $(1+x)^{-1}$ as $1-x+x^2-x^3+\ldots$, ($|x| < 1$), obtain the power series expansion of $\log(1+x)$. 2+2

Deduce that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\log 2$

5. (a) If $f: [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$, then prove that $|f|$ is R-integrable over $[a, b]$ 1+3

and $\left|\int_a^x f(x)dx\right| \le \int_a^b |f(x)|dx$

(b) If $f: [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$. 2+3

For $a \le x \le b$, put $F(x)=\int_a^x f(t)dt$.

Prove the following:

(i) F is of bounded variation on $[a, b]$

(ii) If f is continuous at a point $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

(c) Let $f: [a, b] \to \mathbb{R}$ be defined as follows:
$f(x) = x$, for $x \in [0, 1)$
$= 0$, for $x = 1$.
Find the primitive of f and using fundamental theorem of integral Calculus, establish that $\int_0^1 f(x)dx = 1$. 1+1

6. (a) Use the integral definition of $\log x$ to prove that for $x > 0$, $\log(1+x) > x - \frac{1}{2}x^2$. 3

(b) State the second Mean Value Theorem of integral Calculus in Bonnet's form and use it to prove that 1+3

$$\left|\int_\lambda^\mu \sin^2 x \, dx\right| \le \frac{1}{\lambda}, \text{ if } 0 < \lambda < \mu < \infty.$$

(c) Let $f: [a, b] \to \mathbb{R}$ be an increasing function. Show that f is Riemann integrable over $[a, b]$. 4

7. (a) If f is monotonic on $[a, b]$, then prove that f is of bounded variation on $[a, b]$. 3

(b) Establish by an example that boundedness of f' is not necessary for a function f to be of bounded variation. 3

(c) Let $f(x)$ be a function of period 2π s.t. 5

$f(x) = \frac{x}{2}$ over the interval $0 < x < 2\pi$.

Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\frac{\pi}{2} - [\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots].$$

8. (a) It is given that the following integral converges 4

$I = \int_0^\infty \frac{\log(1+4x^2)}{x^2} dx$. By introducing a parameter in the integrand and carrying a suitable differentiation under the integral sign, show that $I = 2\pi$.

f(x₁-x)

(b) Show that for $p > 1$, the integral $\int_0^\infty \frac{\log x}{x^p}\,dx$ is convergent. 4

(c) Show that the integral $\int_0^1 \frac{1}{\sqrt{x}} \sin\frac{1}{x}\,dx$ is absolutely convergent. 3

9. (a) Using the method of Lagrange's multipliers, find the points on the sphere $x^2 + y^2 + z^2 = 14$ where $3x - 2y + z$ attains its maximum value. 4

(b) Find the radius of convergence of the following series: 3

$$1+\frac{ab}{c}x+\frac{a(a+1)b(b+1)}{2!\ c(c+1)}x^2+\frac{a(a+1)(a+2)\ b(b+1)(b+2)}{3!\ c(c+1)(c+2)}x^3+\ldots..$$

(c) State and prove Mean Value Theorem for a function of two real variables. 1+3

10. (a) Evaluate $\iiint \frac{dx\,dy\,dz}{x^2+y^2+(z-2)^2}$ over the sphere $x^2+y^2+z^2 \le 1$. 4

(b) Find the area of the surface generated by revolving the cardioide $r = a(1+\cos\theta)$ about the initial line. 3

(c) Test whether the graph of the following function is a rectifiable curve 4

$$f(x) = x\cos\frac{\pi}{2x} \quad \text{for } x \neq 0$$
$$= 0 \quad \text{for } x = 0$$

Group-B
[Marks-15]

Answer any *one* question from the following : $15\times1 = 15$

11. (a) Define metric on a set $X\ (\neq \Phi)$. 2+3

Let X be the set of all convergent sequences in $\mathbb{R}$ and $d : X\times X \to \mathbb{R}$ be defined by $d(x, y) = \sup\{|x_n - y_n| : n \in N\}$ for all $x = \{x_n\}$ and $y = \{y_n\}$ in X. Show that (X, d) is a metric space.

(b) Define Cauchy sequence in a metric space (X, d). Prove that a convergent sequence $\{x_n\}$ is (X, d) is a Cauchy sequence. Give an example to show that a Cauchy sequence need not be convergent in an arbitrary metric space. 1+3+1

(c) Let Y be a subspace of a metric space (X, d) then prove that 3+2

(i) $G(\subseteq Y)$ is open in a metric space (Y, d_Y) iff $G = H \cap Y$ for some open set H in (X, d).

(ii) $F(\subseteq Y)$ is closed in (Y, d_Y) iff $F= V \cap Y$ for some closed set V in (X, d).

12. (a) Let (X, d) be a discrete metric space, then prove that any subset of (X, d) is both closed and open. 2+2

(b) Define the term 'complete metric space'. In a metric space (X, d), show that a Cauchy sequence is convergent iff it has a convergent sub sequence.

(c) Define closed set in a metric space (X, d). Show that for any $A \subseteq X$, $A°$ is an open set and $\bar{A}$ is a closed set where $A°$ and $\bar{A}$ denote the interior and closure of A respectively. Give an example of a nested sequence of open intervals in $\mathbb{R}$. 1+2+2+1

Group-C
[Marks-15]

Answer any *one* question from the following : 15×1= 15

13. (a) Find the complex number z that corresponds to the point $\left(\frac{2}{5}, \frac{2\sqrt{3}}{5}, \frac{3}{5}\right)$ on the Riemann sphere $x_1^2 + x_2^2 + x_3^2 = 1$. 3

(b) Let u, v be real valued functions such that $f(x, y) = u(x, y) + i\, v(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then prove that the functions u and v are differentiable at the point (x_0, y_0) and satisfy the Cauchy-Riemann equations. 5

(c) Define harmonic function. Show that the function $u(x, y) = x^3 - 3xy^2$, $x, y \in \mathbb{R}$ is a harmonic function. Apply Milne–Thomson's methods to find a real valued function v such that $u + iv$ is analytic on $\mathbb{C}$. 1+3+3

14. (a) Prove that $f(z)=\begin{cases} \dfrac{z\,\mathrm{Re}\,z}{|z|}, & \text{if } z\neq 0 \\ 0, & \text{if } z=0 \end{cases}$ 2+3

is continuous at $z = 0$ but not differentiable at $z = 0$.

(b) Let $f(z) = u(x, y) + i\,v(x, y)$ where $u(x, y)$ and $v(x, y)$ are real value functions, be defined on a region G except at $z_0 = x_0 + iy_0$, then show that $\lim\limits_{z\to z_0} f(z) = a + ib$ iff $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} u(x, y) = a$ and $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} v(x, y) = b$. 5

(c) Let $\{z_n\}$ be a complex sequence. Show that if $\lim\limits_{n\to\infty} z_n = a$ and $\lim\limits_{n\to\infty} z_n = b$, then $a = b$. 2+2+1

Also show that if $\{z_n\}$ converges to z then $\{|z_n|\}$ converges to $|z|$. Is the converse true? Justify.