MATH·I·15

MTMA (HN)-01

West Bengal State University

B.A./B.Sc./B.Com (Honours, Major, General) Examinations, 2015

PART – I

MATHEMATICS — HONOURS

Paper – I

Duration : 4 Hours] [Full Marks : 100

The figures in the margin indicate full marks.

GROUP - A

Answer any *five* questions. $5 \times 5 = 25$

1. i) If a, b, c are three positive integers such that $gcd(a, c) = 1$ and c divides ab then prove that c divides b. 2

   ii) Use mathematical induction to prove that for any positive integer n.
   $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^2 + \ldots + n \cdot 2^n = (n-1)2^{n+1} + 2.$ 3

2. i) Prove that $3 \cdot 4^{n+1} \equiv 3 \pmod 9$ for all positive integers n. 2

   ii) If n is an even positive integer then prove that $\phi(2n) = 2\phi(n)$, where ϕ stands for Euler's phi-function. 3

3. i) By Fermat's theorem, show that $a^{12} - b^{12}$ is divisible by 91 if a and b are both prime to 91. 3

   ii) If $d = gcd(a, b)$, then show that $gcd(a^2, b^2) = d^2$. 2

4. Show that the principal value of the ratio of $(1+i)^{1-i}$ and $(1-i)^{1+i}$ is $\sin(\log 2) + i\cos(\log 2)$. 5

5. Prove that $\sin\left[i \log \frac{a-ib}{a+ib}\right] = \frac{2ab}{a^2+b^2}$.

6. i) Solve the equation : $3z^5 + 2 = 0$.

ii) If $x = \cos\theta + i\sin\theta$, $y = \cos\phi + i\sin\phi$ and m and n are integers then prove that $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$.

7. i) Find the condition that the roots of $x^4 + px^3 + qx^2 + rx + s = 0$ may be in G.P.

ii) If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find the value of $\sum \frac{1}{\alpha^2\beta}$.

8. i) Find the number and position of the real roots of the equation $x^4 - 6x^3 + 10x^2 - 6x + 1 = 0$.

ii) Prove that $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ cannot have a multiple root.

9. i) Find the equation whose roots are the roots of the equation $x^4 - 8x^2 + 8x + 6 = 0$, each diminished by 2.

ii) Solve the equation $16x^4 - 64x^3 + 56x^2 + 16x - 15 = 0$ whose roots are in A.P.

GROUP - B

Answer any *two* questions. $2 \times 10 = 20$

10. i) Suppose that P and Q are two sets. R is such a set that contains elements belonging to P or Q but not both. T is such a set that contains elements belonging to Q or complement of P but not both. Show that R is the complement of T. 3

ii) Define a partition of a set. Show that an equivalence relation partitions a set into disjoint equivalence classes. 1 + 3

iii) Show that a mapping $f : A \to B$ is right invertible if and only if it is surjective. 3

11. i) Prove or set counter example to disprove the statement — 'The union of two equivalence relations is again an equivalence relation'. 2

ii) Let $f : A \to B$ and $g : B \to C$ be two mappings such that $g \, o \, f : A \to C$ is injective. Is it necessary that g is injective ? Justify your answer. 3

iii) Prove that a semigroup is a group if it is a quasigroup. 5

12. i) Give an example of a group of non-prime order, whose each element other than the identity has the same order. 3

ii) Let (G, o) be a group such that $(a \, o \, b)^n = a^n \, o \, b^n$ for any three consecutive positive integral values of n and for all $a, b \in G$. Prove that G is a commutative group. 4

iii) Let G be the group of all non-zero rational numbers under multiplication. Examine whether $H = \left\{ \dfrac{1+3n}{1+3m} : n, m \in \mathbf{Z} \right\}$ is a subgroup of G. 3

13. i) Define a Boolean ring. Prove that in a Boolean ring R, $2a = 0$ for all $a \in R$. Hence show that every Boolean ring is a commutative ring. 5

ii) Prove that the set $R = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \right\}$ forms a field with respect to usual matrix addition and multiplication. 5

GROUP - C

Answer any *three* questions. $3 \times 5 = 15$

14. If A be a skew-symmetric matrix and $(I + A)$ be a non-singular matrix, then show that $B = (I - A) (I + A)^{-1}$ is orthogonal. 5

15. Compute the inverse of the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 2 & -1 \\ 7 & 3 & 1 \end{pmatrix}$ by using row operations. 5

16. Find the non-singular matrices P and Q, such that PAQ is in the normal form and hence find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$. 4 + 1

17. If the system of equations

$$(f^2 - bc)x + (ch - fg)y + (bg - hf)z = 0$$

$$(ch - fg)x + (g^2 - ca)y + (af - gh)z = 0$$

$$(bg - hf)x + (af - gh)y + (h^2 - ab)z = 0$$

has a solution other than $x = y = z = 0$, then show that

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$ 5

18. If $\sum_{i=1}^{4} a_i^2 = \sum_{i=1}^{4} b_i^2 = \sum_{i=1}^{4} c_i^2 = \sum_{i=1}^{4} d_i^2 = 1$ and

$$\sum_{i=1}^{4} a_i b_i = \sum_{i=1}^{4} b_i c_i = \sum_{i=1}^{4} c_i d_i = \sum_{i=1}^{4} a_i c_i = \sum_{i=1}^{4} a_i d_i = \sum_{i=1}^{4} b_i d_i = 0$$

then show that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \pm 1.$$ 5

19. Reduce the quadratic form $6x^2 + y^2 + 18z^2 - 4yz - 12zx$ to its normal form and then examine whether the quadratic form is positive definite or not. 4 + 1

GROUP - D

Answer any *one* question. $1 \times 10 = 10$

20. i) A farmer has a 100 acre farm. He can sell tomatoes, lettuce and radishes and can raise the price to obtain Re. 1.00 per kg for tomatoes, Re. 0.75 a head for lettuce and Rs. 2.00 per kg for radishes. The average yield per acre is 2000 kg of tomatoes, 3000 heads of lettuce and 1000 kg of radishes. Fertilizers are available at Re. 0.50 per kg and the amount required per acre is 100 kg for each tomato and lettuce and 50 kg for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for each tomato and radish and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20.00 per man-day. Formulate this problem as an LP model to maximize the farmers profit. 5

ii) Solve the Linear Programming Problem by graphical method :

Min $Z = 20x + 40y$

subject to

$36x + 6y \geq 108$

$20x + 10y \geq 100$

$3x + 12y \geq 36$, and $x \geq 0$ and $y \geq 0$. 5

21. i) Find all basic feasible solutions of the system of equations :

$$3x_1 + 5x_2 - 7x_3 = 21$$

$$6x_1 + 10x_2 + 3x_3 = 42$$

Which of these are degenerate ? 4

ii) Write the following L.P.P. :

Maximize $Z = 2x_1 + 3x_2$

subject to
$$x_1 + x_2 \le 30$$
$$x_2 \ge 3$$
$$x_2 \le 12$$
$$x_1 \le 20$$
$$x_1 - x_2 \ge 0$$
$$x_1, x_2 \ge 0$$

in the matrix form. 3

iii) Verify whether the feasible solution $x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 2$ to the system
$$x_1 + x_2 + 5x_3 = 2$$
$$x_1 + x_2 - 31x_3 = 2$$
$$2x_1 + 4x_2 + 13x_3 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \ge 0$$

is basic or not. 3

GROUP - E

SECTION - I

Answer any *three* questions. $3 \times 5 = 15$

22. Reducing the equation $x^2 + 2xy + y^2 - 4x + 8y - 6 = 0$ to its Canonical form, show that it represents a parabola. Find the directrix of the parabola. 3 + 2

23. If the normal be drawn at one extremity $(l, \frac{\pi}{2})$ of the latus rectum PSP' of the conic $\frac{l}{r} = 1 + e\cos\theta$ where S is the pole, then show that the distance from the focus S of the other point in which the normal meets the conic is $\frac{l(1 + 3e^2 + e^4)}{1 + e^2 - e^4}$. 5

24. If the straight lines $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and the straight line $lx + my = 1$ be one of its diagonals, then show that the equation of the other diagonal is $y (bl - hm) = x (am - hl)$. 5

25. Prove that the equation

$$(ab - h^2)(ax^2 + 2hxy + by^2 + 2gx + 2fy) + af^2 + bg^2 - 2fgh = 0$$

represents a pair of straight lines. Prove also that these lines form a rhombus with the straight lines $ax^2 + 2hxy + by^2 = 0$, if $(a - b) fg + h (f^2 - g^2) = 0$.

2 + 3

26. i) Prove that the straight lines joining the origin to the points of intersection of the straight line $\frac{x}{\alpha} + \frac{y}{\beta} = 2$ and the circle $(x - \alpha)^2 + (y - \beta)^2 = \gamma^2$ are at right angles, if $\alpha^2 + \beta^2 = \gamma^2$. 2

ii) Show that the straight line $\frac{l}{r} = A \cos \theta + B \sin \theta$ touches the conic $\frac{l}{r} = 1 + e \cos \theta$, if $(A - e)^2 + B^2 = 1$. 3

SECTION - II

Answer any *three* questions. $3 \times 5 = 15$

27. i) Find the foot of the perpendicular drawn from the point $P (1, 8, 4)$ on the straight line joining the points $A (0, - 11, 4)$ and $B (2, - 3, 1)$. 3

ii) Find the points on the x-axis whose distance from the point $(2, - 2, 4)$ is 6 units. 2

28. Find the equation of the plane which passes through the point $(2, 1, - 1)$ and is orthogonal to each of the planes $x - y + z = 1$ and $3x + 4y - 2z = 0$. 5

29. The plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α. Show that the equation of the plane in the new position is

$$ax + by \pm z \sqrt{a^2 + b^2} \tan \alpha = 0 .$$ 5

30. A variable straight line always intersects the lines $x = k, \ y = 0$; $y = k, \ z = 0$; $z = k, \ x = 0$. Prove that the equation to its locus is

$$xy + yz + zx - k (x + y + z - k) = 0.$$ 5

31. Show that the equation to the plane containing the straight line $\frac{y}{b} + \frac{z}{c} = 1, \ x = 0$ and parallel to the straight line $\frac{x}{a} - \frac{z}{c} = 1, \ y = 0$ is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$ and if $2d$ be the shortest distance between the lines, then show that

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$ 3 + 2