West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2014

PART - III

MATHEMATICS – Honours

PAPER –V

Duration : 4 Hours Maximum Marks : 100

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

GROUP - A

(Marks : 70)

Answer question No. 1 and any *five* from the rest.

1. Answer any *five* of the following : $5 \times 3 = 15$

a) Prove or disprove :The range of any convergent sequence in $\mathbb{R}$ is a compact set.

b) Prove that the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = x^4 + x^3$ is of bounded variation over $[-1, 1]$.

c) If e is defined by $\int_1^e \frac{dt}{t} = 1$, prove that $2 < e < 3$.

d) Test the convergence of $\int_1^\infty \frac{\cos ax - \cos bx}{x} dx$; $a, b \in \mathbb{R} - \{0\}$.

Check whether true or false : There exists no power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence 1 which is not convergent at both $x = \pm 1$.

Let $x : [\ 0,\ 1\ ] \to I\!R$ and $y : [\ 0,\ 1\ ] \to I\!R$ be defined by

$$x(t) = \frac{1}{3^n},\ \frac{1}{3^{n+1}} < t < \frac{1}{3^n};\ n = 0, 1, 2, \ldots..$$
$$= 0,\ t = 0$$

and
$$y(t) = t^3 \sin \frac{1}{t^2},\ t \neq 0$$
$$= 0,\ t = 0$$

Prove that the curve $\gamma = (\ x, y\ )$ is rectifiable.

g) Determine the perimeter of the cardioid $r = a\ (\ 1 + \cos\theta\ ),\ a > 0$.

h) Evaluate $\iint \sqrt{4a^2 - x^2 - y^2}\ dx\ dy$ taken over the upper half of the circle $x^2 + y^2 - 2ax = 0$.

i) Prove or disprove : The trigonometric series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ represents a Fourier series.

a) Prove that a compact subset of $I\!R$ is closed and bounded in $I\!R$. 5

b) Discuss whether the set $\left\{ 1 + \frac{(-1)^n}{n} \right\}$ is compact or not. 3

c) Every point of a compact set S in $I\!R$ is an isolated point of S. Prove that S is finite. 3

a) State and prove Taylor's theorem for a real-valued function of two independent variables. 1 + 3

b) If $f\ (\ x\ ,\ y\ ) = \sin \pi x + \cos \pi y$, use Mean value theorem to express $f\left(\frac{1}{2}, 0\right) - f\left(0, -\frac{1}{2}\right)$ in terms of the first order partial derivatives of f and deduce that there exists θ in $(\ 0,\ 1\ )$ such that

$$\frac{4}{\pi} = \cos \frac{\pi}{2}\theta + \sin \frac{\pi}{2}(1 - \theta).$$ 3

c) Prove that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$. 4

4. a) Let $f : [a, b] \to I\!R$ be a bounded function on $[a, b]$. Prove that to each positive ε there corresponds a positive δ such that $\underline{\int_a^b} f - \varepsilon < L(P, f)$ for all partitions P of $[a, b]$ satisfying $\|P\| \le \delta$.

Further if $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that $\|P_n\| \to 0$ as $n \to \infty$ then prove that

$$\lim_{n \to \infty} L(P_n, f) = \underline{\int_a^b} f.$$ 3 + 2

b) Prove that $f : [0, 1] \to I\!R$ defined by $f(0) = 0$ and $f(x) = \left[x^{-1}\right]^{-1}$, for $0 < x \le 1$ is integrable, where $[y]$ denotes the integral part of y. 2

c) A function $f : [0, 1] \to I\!R$ is defined by

$$f(x) = x + x^3, \text{ if } x \text{ is rational}$$
$$= x^2 + x^3, \text{ if } x \text{ is irrational.}$$

Evaluating $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$, examine the integrability of f in $[0, 1]$. 4

5. a) If $f : [a, b] \to I\!R$ be integrable on $[a, b]$ then prove that the function $F : [a, b] \to I\!R$ defined by $F(x) = \int_a^x f(t)\, dt$ is continuous on $[a, b]$.

Give an example to show that continuity of f is not needed for the continuity of F. 3 + 2

b) Prove that for $0 < c < 1$, $2 \int_0^1 \sqrt{1 - c^2 \sin^2 x}\, dx > \sqrt{1 - c^2} + 1$. 3

c) Let $f(x) = x[x]$, $x \in [0, 3]$. Show that f is integrable on $[0, 3]$. Further show that evaluation of $\int_0^3 f$, cannot be done by the fundamental theorem of Integral calculus. 3

a) Show that the integral $\int_0^1 x^{p-1} \log x\, dx$ is convergent if and only if $p > 0$. 4

b) Test the convergence of the integral $\int_0^1 \sin x^{1/3} \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$. 3

c) If f is continuous on $[0, 1]$, show that $\int_0^1 \frac{f(x)}{\sqrt{1 - x^2}}\, dx$ is convergent. 4

a) Define uniform convergence of a sequence of functions $\{f_n\}_n$ on $S\ (\subset \mathbb{R})$.

A sequence of functions $\{f_n\}$ is uniformly convergent on $S\ (\subset \mathbb{R})$ to a function f and $\lim_{x \to x_0} f_n(x) = a_n$, where $x_0 \in S'$. Prove that the sequence $\{a_n\}$ is convergent and $\lim_{x \to x_0} f(x)$ exists and is equal to $\lim_{n \to \infty} a_n$. 1 + 4

b) For a sequence of functions $\{f_n\}$, where each f_n is differentiable on $[a, b]$, does uniform convergence of $\{f_n\}$ implies uniform convergence of $\{f_n'\}$? Justify your answer. 3

c) Examine uniform convergence of $\{f_n\}$ on $[0, 1]$, where $f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$, $x \in [0, 1]$, for each $n \in \mathbb{N}$. 3

8. a) Prove that the series $\sum_{n=0}^{\infty} (-1)^n \dfrac{\cos nx}{p^2 - n^2}$ is uniformly convergent on any closed and bounded interval $[a, b]$, where $p \neq 0, \pm 1, \pm 2, \ldots\ldots$. 4

b) Let $\sum_{1}^{\infty} f_n(x)$ be uniformly convergent to $f(x)$ on $[a, b]$, where each f_n is continuous on $[a, b]$ and let $g : [a, b] \to I\!R$ be integrable on $[a, b]$; prove that $\int_a^b f(x)\, g(x)\, dx = \sum_{1}^{\infty} \int_a^b f_n(x)\, g(x)\, dx$. 3

c) A series $\sum_{n=1}^{\infty} f_n(x)$ of differentiable functions f_n on $[0, 1]$ is such that

$$S_n(x) = \sum_{i=1}^{n} f_i(x) = \frac{\log\left(1 + n^4 x^2\right)}{2n^2},\ x \in [0, 1] \text{ and } n \in N. \text{ Show that}$$

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x),\ x \in [0, 1].$$

What can be said about uniform convergence of $\sum_{1}^{\infty} f_n'(x),\ x \in [0, 1]$? 4

9. a) If $\sum_{0}^{\infty} a_n x^n$ be a power series with radius of convergence $R\ (>0)$ and $\sum_{0}^{\infty} a_n R^n$ is convergent then prove that $\sum_{0}^{n} a_n x^n$ is uniformly convergent on $[0, R]$.

Further if the sum of $\sum_{0}^{\infty} a_n x^n$ be $f(x)$ on $(-R, R)$, prove that $\sum_{n=0}^{\infty} a_n R^n = \lim_{x \to R-} f(x)$. 4 + 2

b) From the relation $\sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$, $|x| < 1$; show that $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \ldots\ldots, |x| \le 1$. Hence deduce that $\frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \ldots\ldots$. 4 + 1

10\. a) Obtain the Fourier series of the function $f : [-\pi, \pi] \to I\!R$ defined as

$$f(x) = \cos x,\ 0 \le x \le \pi$$
$$= -\cos x,\ -\pi \le x < 0.$$

Hence find the sum of the series $\frac{2}{1.3} - \frac{6}{5.7} + \frac{10}{9.11} - \ldots\ldots$ 3 + 1

b) Show that $\int_0^{\pi/2} \log\left(\frac{a + b \sin\theta}{a - b \sin\theta}\right) \frac{1}{\sin\theta}\, d\theta = \pi \sin^{-1} \frac{b}{a};\ a > b \ge 0.$ 4

c) Show that $\iint_E y^2 \sqrt{a^2 - x^2}\, dx\, dy = \frac{32}{45} a^5$, where $E = \{(x, y) \in I\!R^2 : x^2 + y^2 \le a^2\}$. 3

GROUP - B

(Marks : 15)

Answer any *one* of the following.

11\. a) Let $C[0, 1]$ denotes the set of all real valued continuous functions on $[0, 1]$. For $x, y \in [0, 1]$, let $d(x, y) = \int_0^1 |x(t) - y(t)|\, dt$. Show that d is a metric on $C[0, 1]$. 4

b) A subset of a metric space (X, d) is said to be closed if $X - A$ is an open set. Show that an arbitrary union of open sets is an open set and an arbitrary intersection of closed sets is a closed set. 3 + 3

c) If $\{x_n\}, \{y_n\}$ are convergent sequences in a metric space (X, d), show that $\lim\limits_{n \to \infty} d(x_n, y_n)$ exists.

Hence correct or justify : $d(x, y)$ is a continuous function. 3 + 2

12. a) Let X denotes the set of all real valued sequences, and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad \text{where} \quad x = \{x_n\} \quad \text{and} \quad y = \{y_n\}.$$

Show that d is a metric. 4

b) Define Cauchy sequence in a metric space (X, d). Prove that a Cauchy sequence in a metric space is convergent *iff* it has a convergent subsequence. Define the term 'complete metric space'. 1 + 4 + 1

c) Let $C[a, b]$ denotes the metric space of all continuous functions defined on closed bounded interval $[a, b]$ with the usual metric $d(x, y) = \sup\limits_{a \le t \le b} |x(t) - y(t)|$.

Show that in this metric space a sequence $\{x_n\}$ converges to x *iff* $\{x_n(t)\}$ converges uniformly to $x(t)$ on $[a, b]$. 5

GROUP - C

(Marks : 15)

Answer any *one* of the following.

13. a) If Z is a point on the complex plane and (α, β, γ) is the stereographic projection on the Riemann sphere $x^2 + y^2 + \left(Z - \frac{1}{2}\right)^2 = \frac{1}{4}$ then prove that $Z = \frac{\alpha + i\beta}{1 - \gamma}$ $(i = \sqrt{-1})$. 5

b) Let u, v be real-valued functions such that $f(x+iy) = u(x, y) + iv(x, y)$ is differentiable at $Z_0 = x_0 + iy_0$. Then prove that the function u and v are differentiable at the point (x_0, y_0) and satisfy the Cauchy-Riemann equation. 5

c) Prove that $u(x, y) = x^3 - 3xy^2\ [x, y \in R]$ is a harmonic function on $\mathbb{R}^2$. Find the harmonic conjugate of u. 5

a) Prove that the function

$$f(x+iy) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0 \end{cases}$$

Satisfies Cauchy-Riemann equations at the origin but $f'(0)$ does not exist. 5

b) Find the points where the following function f is differentiable and hence, deduce that it is nowhere analytic : $f = u + iv$, where $u(x, y) = x^2y^2$, $v(x, y) = 2x^2y^2$. 5

c) Prove that a function $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ is continuous at $Z_0 \in D$ if and only if $f(Z_0) = \lim_{n \to \infty} f(Z_n)$ whenever $Z_n \to Z_0$ as $x \to \infty$, $Z_n \in D$, for $n \in \mathbb{N}$. 5