West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2013

PART – II

MATHEMATICS — HONOURS

Paper – IV

Duration : 4 Hours] [Full Marks :

The figures in the margin indicate full marks.

GROUP – A

Answer any *two* questions. $2 \times 10 =$

1. a) The tangents at two points P and Q of a parabola, whose focus is S meet a

Show that $SP.SQ = ST^2$.

b) Prove that the locus of the poles of the chord of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ wh

subtends a right angle at the centre is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$.

2 a) Two cones are described with guiding curves $xz = a^2$, $y = 0$; $yz = b^2$, $x = 0$

with any common vertex. If their four common generators meet the place z

in four concyclic points, then show that the vertex lies on the surf

$z(x^2 + y^2) = a^2 x + b^2 y$.

b) Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the lines $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$ meet the plane $z=0$ in circles. 5

a) Show that the perpendiculars from the origin on the generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ lies on the cone $\left(\frac{x}{a}-\frac{y}{b}\right)(ax-by)+2z^2=0$. 5

b) Reduce the equation $2x^2+20y^2+18z^2-12yz+12xy+2zx+6y-2z=2$ to its canonical form and identify the surface it represents. 5

GROUP – B

Answer any *one* question. 1 × 10 = 10

a) Find the eigenvalues and the corresponding eigenfunctions of the boundary value problem $y''+2y'+(1+\lambda)y=0$, $y(0)=0$, $y'(a)=0$. 5

b) Solve $\frac{dx}{x(x+y)}=\frac{dy}{-y(x+y)}=\frac{dz}{-(x-y)(2x+2y+z)}$. 5

a) Form partial differential equation by eliminating arbitrary functions f and g from $z=f(x^2-y)+g(x^2+y)$. 5

b) Find a complete integral of $(p+q)(px+qy)=1$ by Charpit's method. 5

GROUP – C

Answer *either* Q. No. 6 *or* Q. No. 7 and *either* Q. No. 8 *or* Q. No. 9.

13 + 12 =

6. a) Prove that the set of all convex combination of a finite number of points convex set.

b) Use the penalty method to solve the following L.P.P. :

Maximize $Z = 2x_1 + x_2 + 3x_3$

subject to $x_1 + x_2 + 2x_3 \geq 5$

$$2x_1 + 3x_2 + 4x_3 = 12$$

$$x_1, x_2, x_3 \geq 0$$

7. a) Use Duality to solve the following L.P.P. :

Maximize $Z = 2x_1 + x_2$

subject to the constraints $x_1 + 2x_2 \leq 10$

$$x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

$$x_1 - 2x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

b) If any iteration of the simplex algorithm $z_j - c_j < 0$ for at least one j for w $y_{ij} \leq 0 (i = 1, 2, ..., m)$, then there does not exist any optimum solution to L.P.P.

8. a) Consider the following transportation problem :

Factory	Godowns						Stock available
	1	2	3	4	5	6	
A	7	5	7	7	5	3	60
B	9	11	6	11	—	5	20
C	11	10	6	2	2	8	90
D	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	

It is not possible to transport any quantity from factory *B* to Godown 5. Determine (i) Initial solution by VAM method, (ii) Optimum basic feasible solution. 6

b) A departmental head has four subordinates, and four tasks to be performed. The subordinate differ in efficiency, and the tasks differ in intrinsic difficulty. His estimate, of the time each man would take to perform each task, is given in the matrix below :

Tasks	Men			
	E	*F*	*G*	*H*
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man-hours ? 6

9. a) Solve graphically the following game problem : 6

		B_1	B_2	B_3
A	A_1	1	3	11
	A_2	8	5	2

b) Solve the following game problem by converting it into L.P.P. : 6

		Q_1	Q_2	Q_3
Player P	P_1	9	1	4
	P_2	0	6	3
	P_3	5	2	8

GROUP – D

Answer any *three* questions. $3 \times 15 = 45$

10. a) A particle moves towards a centre of force, the acceleration at a distance x being given by $\mu\left(x + \frac{a^4}{x^3}\right)$; if it starts from rest at a distance a, show that it will arrive at the centre in time $\frac{t}{4\sqrt{\mu}}$. 7

b) A particle is projected upwards with a velocity U in a medium whose resistance varies as the square of the velocity. Prove that it will return to the point of projection with velocity $v = \frac{UV}{\sqrt{U^2 + V^2}}$ after a time $\frac{V}{g}\left\{\tan^{-1}\frac{U}{V} + \tan h^{-1}\frac{v}{V}\right\}$ where V is the terminal velocity. 8

11. a) A particle is projected with velocity u at an inclination α above the horizontal in a medium whose resistance per unit mass is k times the velocity. Show that its direction will again make angle α below the horizontal after a time $\frac{1}{k}\log\left(1+\frac{2ku}{g}\sin\alpha\right)$. 7

b) A ball is thrown from a point on a smooth horizontal ground with a velocity V at an angle α to the horizon. Assuming e to be the coefficient of restitution of the ball with the ground, show that the total time for which the ball rebounds on the ground is $\frac{2V\sin\alpha}{g(1-e)}$, and that its distance from the starting point when it ceases to rebound is $\frac{V^2\sin 2\alpha}{g(1-e)}$. 8

12. a) A particle describes a plane curve under an acceleration which is always directed towards a fixed point (i.e. under the action of a central force P per unit mass); Show that $\frac{h^2}{p^3}.\frac{\mathrm{d}p}{\mathrm{d}r}=P$. 7

b) A particle is placed at rest in a rough tube at a distance a from one end and the tube starts rotating horizontally with a uniform angular velocity w about this end. Show that the distance of the particle at time t is $ae^{-wt}\tan\in[\cos h(wt\sec\in)+\sin\in\sin h(wt\sec\in)]$ where $\tan\in$ is the coefficient of friction. 8

13. a) A planet is describing an ellipse about the sun as focus. Show that its velocity away from the sun is greater when the radius vector to the planet is at right angle to the major axis of the path and that, then it is $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where $2a$ is the major axis, e the eccentricity, and T the periodic time. 7

b) If a rocket, originally of mass M, throws off every unit of time a mass eM with relative velocity V, and if M' be the mass of the case etc, show that it cannot rise at once unless $eV > g$, not at all unless $\frac{eMV}{M'} > g$. It just rises vertically at once, show that its greatest velocity is $V \log \frac{M}{M'} - \frac{g}{e}\left(1 - \frac{M'}{M}\right)$ and that the greatest height it reaches is $\frac{V^2}{2g}\left(\log \frac{M}{M'}\right)^2 + \frac{V}{e}\left(1 - \frac{M'}{M} - \log \frac{M}{M'}\right)$

(Here M includes M').

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Part – III

MATHEMATICS — HONOURS

PAPER – V

Duration : 4 Hours [Maximum Marks : 100

The figures in the margin indicate full marks.

GROUP - A

(Marks — 70)

Answer question No. 1 and any *five* from the rest.

1. Answer any *five* of the following questions : $5 \times 3 = 15$

a) Prove that a non-empty compact set of real numbers has a greatest element.

b) A function $f : [\,0,\ 1\,] \to I\!R$ is defined by $f(x) = x^2 - x - 18$. Show that f is a function of bounded variation on $[\,0,\ 1\,]$ and find $V_f[\,0,\ 1\,]$.

c) From the definition $\log_e x = \int_1^x \frac{dt}{t}$, $x > 0$; prove that $\frac{x}{1+x} < \log_e(1+x) < x$, for $x > 0$.

d) Applying Dirichlet's test determine the convergence of $\int_0^\infty \frac{\sin x}{x}\,dx$.

e) A sequence of real valued functions $\{f_n\}_n$ defined on $(-\infty,\ \infty)$ converges uniformly to a continuous function f on $(-\infty,\ \infty)$. Prove that $\lim_{n \to \infty} f_n\left(x + \frac{1}{n}\right) = f(x),\ -\infty < x < \infty$.

f) Is the power series $\sum_{n=0}^{\infty} \frac{e^n}{(2n)!} x^n$ convergent everywhere ?

g) Prove that the series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots, x \in [0, 1]$ is uniformly convergent on $[0, 1]$.

h) Let $f(x, y) = \sin x \cos y$, $(x, y) \in \mathbb{R}^2$. Applying mean vale theorem prove $\frac{\pi}{3} \cos \frac{\pi\theta}{3} . \cos \frac{\pi\theta}{6} - \frac{\pi}{6} \sin \frac{\pi\theta}{3} \sin \frac{\pi\theta}{6} = \frac{3}{4}$, $\theta \in (0, 1)$.

i) Find the length of the curve $x = e^{\theta} \left(\sin \frac{\theta}{2} + 2\cos \right.$ $y = e^{\theta} \left(\cos \frac{\theta}{2} - 2\sin \frac{\theta}{2} \right)$ measured from $\theta = 0$ to $\theta = \psi$.

2. a) Prove that every infinite subset of a compact set A in $\mathbb{R}$ has a limit point in A

b) The functions $f : D \to \mathbb{R}$ is continuous, and $D (\subset \mathbb{R})$ is compact in $\mathbb{R}$, prove $f(D)$ is compact in $\mathbb{R}$.

c) Using definition, prove that $(0, 1]$ is not a compact set.

3. a) Applying Taylor's theorem for a homogeneous function $f(x, y)$, $(x, y) \in \mathbb{R}$ degree n, having continuous second order partial derivatives, prove $x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f(x, y)$.

b) Let $u = a^3 x^2 + b^3 y^2 + c^3 z^2$, where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Apply Lagrange's me of undetermined multiplier to find the stationary value of u.

c) A function $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = x \sin \frac{\pi}{x}, \; 0 < x \leq 1$$

$$= 0, \; x = 0.$$

Show that f is not of bounded variation on $[0, 1]$.

4. a) Prove that a continuous function on $[a, b]$, $a, b \in \mathbb{R}$; is Riemann integrable thereon. 3

b) If $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and possesses a primitive ϕ on $[a, b]$, then prove that $\int_a^b f = \phi(b) - \phi(a)$. Give an example of a function f which has a primitive but the function f is not integrable. 3 + 2

c) For $x \geq 0$, $\phi(x) = \lim_{n \to \infty} \frac{x^n + 2}{x^n + 1}$; and $f(x) = \int_0^x \phi(t)\, dt$; show that f is continuous at $x = 1$ but not differentiable at $x = 1$. 3

5. a) A function f is defined on $[0, 1]$ by

$$f(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \quad (n = 0, 1, 2, \ldots..)$$

$$= 0, \quad x = 0.$$

Prove that f is integrable on $[0, 1]$ and find $\int_0^1 f$. 4

b) State Second Mean Value theorem of integral calculus in Bonnet's form. Using it show that $\left| \int_a^b \cos x^2 \, dx \right| \leq \frac{1}{a}$, $0 < a < b < \infty$. 2 + 2

c) If a function f is continuous on $[a, b]$ and $\int_a^b f.g = 0$ for every continuous function g on $[a, b]$, prove that $f(x) = 0$, for all $x \in [a, b]$. 3

6. a) f and g are two positive-valued functions on $[a, X]$ and $\int_a^X f\, dx$, $\int_a^X g\, dx$ exist for all $X > a > 0$. If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$, where l is a non-zero finite number, prove that $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ converge or diverge together. 4

b) Show that the integral $\int_1^\infty x^p \frac{x + \sin x}{x - \sin x}\, dx$ is convergent only when $p < -1$. 3

c) Examine the convergence of the integral $\int_0^\pi \frac{dx}{\cos \alpha - \cos x}$, $0 \leq \alpha \leq \pi$. 4

7\. a) For each $n \in I\!N$, $f_n : D \to I\!R$ is continuous on $D\ (\subset I\!R\ )$. If the sequence $\{f_n\}$ be uniformly convergent on D to a function f, then prove that f is continuous on D.

b) For each $n, \in I\!N$, $f_n(x) = nx,\ 0 \le x \le \frac{1}{n}$

$$= 1,\ \frac{1}{n} < x \le 1$$

Show that the sequence $\{f_n\}$ converges to a function f on $[\ 0,\ 1\ ]$ but the convergence is not uniform.

c) For $n \in I\!N$, $f_n(x) = x^2 e^{-nx}$, $x \in [\ 0, \infty\ )$, show that the sequence $\{f_n\}$ is uniformly convergent on $[\ 0, \infty\ )$.

8\. a) For $n \in I\!N$, $f_n : [a, b] \to I\!R$ is integrable on $[\ a, b\ ]$. If the series $\sum f_n$ be uniformly convergent on $[\ a, b\ ]$ to a function S then prove that S is integrable on $[\ a, b\ ]$ and $\sum \int_a^b f_n(x)\, dx = \int_a^b S(x)\, dx$.

b) Show that $\sum_{n=1}^{\infty} \frac{n^5+1}{n^7+3}\left(\frac{x}{3}\right)^n$ is uniformly convergent on $[\ -3, 3\ ]$.

c) Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^p\left(1+x^n\right)}$ converges uniformly for all $p\ (>0\ )$ on $[\ 0,\ 1\ ]$.

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1. If $\sum_{n=0}^{\infty} a_n$ be convergent, prove that the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on [0, 1]. 4

Determine the interval of convergence of the power series $1 + \frac{x}{2} + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4$ 3

If $f(x) = \{\pi - |x|\}^2$ on $[-\pi, \pi]$, prove that the Fourier series of f is given by $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$. 4

Stating the reasons for validity of differentiation under the sign of integration, prove that for $\alpha < 1$,

$$\int_0^{\pi} \log(1 + \alpha \cos x)\, dx = \pi \log\left(\frac{1 + \sqrt{1 - \alpha^2}}{2}\right).$$ 4

Evaluate $\iint_E \left(x^2 - y^2\right) dx\, dy$ where E is the region bounded by $x - y = 0$, $y = 0$, $x = 1$. 3

Transform the integral $\int_0^{\pi/2}\int_0^{\pi/2} \sqrt{\frac{\sin\phi}{\sin\theta}}\, d\phi\, d\theta$ by the substitution $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$ and find its value. 4

GROUP - B

(Marks — 15)

Answer any *one* of the following.

11. a) i) Define a metric on a set X.

If x, y, z be any three points on a metric space (X, d), then show th

$d(x, y) \geq |d(x, z) - d(z, y)|$.

ii) In every metric space (X, d) show that every open ball in an open set.

3

b) Let X be any metric space. For any two subsets A, B of X show th $D(A \cup B) = D(A) \cup D(B)$ where $D(A)$ etc. are derived set of A.

c) Let (X, d) be a complete metric space and $\{F_n\}$ be a decreasing sequence non-empty bounded closed sets in (X, d) such that $d(F_n) \to 0$ as $n \to$ Prove that $\bigcap_{n=1}^{\infty} F_n$ is a singleton set.

12. a) i) Let X be a non-empty set and consider the discrete metric d be defin on X. Show that all convergent sequence in (X, d) are eventua constant sequences.

ii) Prove that any subset of a discrete metric space is open and closed. 3

b) State and prove Cantor's intersection theorem in a metric space (X, d).

c) i) Let (x, d) be a metric space and $A \subset X$. Then show that if $x \notin A$ and x a limit piont of A then $d(x, A) = 0$.

ii) In a metric space (X, d) if $a, b \in X$ and $a \neq b$, then show that there ex open ball S_a and S_b containing a and b respectively such th $S_a \cap S_b = \phi$. 2

GROUP - C

(Marks — 15)

Answer any *one* of the following.

a) If (x_1, x_2, x_3) is the projection on the Riemann sphere $(x_1^2 + y_1^2 + z_1^2 = 1)$ of the point $z = (x + iy)$ in the complex plane, then show that $x_1 = \dfrac{z + \bar{z}}{|z|^2 + 1}$, $x_2 = \dfrac{z - \bar{z}}{i\left(|z|^2 + 1\right)}$, $x_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$ 5

b) Define analytic function. Construct the analytic function

$f(z) = u(x, y) + iv(x, y)\ (x, y \in R)\ (z = x + iy)$

where $u(x, y) = \dfrac{1}{2}\log\left(x^2 + y^2\right)$ and $f(1) = 0$ and v is a real valued function of x and y. 5

c) $f(x + iy) = u(x, y) + iv(x, y)$ be defined in a region G. Let $u(x, y)$ and $v(x, y)$ be differentiable at (x_0, y_0) and let the Cauchy-Riemann equations be satisfied at (x_0, y_0), prove that f is differentiable at $z_0 = x_0 + iy_0$. 5

4. a) Let $f(z) = u(x, y) + iv(x, y)\ (x, y \in R)\ (z = x + iy)$ where $f : G \to \mathbb{C}$. Prove that f is continuous on G iff both $u(x, y)$ and $v(x, y)$ are continuous on G. 5

b) If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$ find $f(z)$ in terms of z by Milne-Thomson method. 5

c) Prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ where $f(z)$ is an analytic function of z. 5